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K-theory theorems in topological cyclic homology

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Abstract

Topological cyclic homology serves as an approximation to algebraic K-theory. It is more accessible to calculations, but how well does it reflect the structural properties of K-theory? In addition to confinality, resolution and finite products which have been previously discussed for topological Hochschild homology, this paper addresses localization and Devissage. Its main result is a Devissage theorem and a "vanishing of nil-terms" result for topological Hochschild homology and topological cyclic homology. © 1998 Elsevier Science B.V. All rights reserved.

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Topological cyclic homology (TC) was developed in order to have an approximation of algebraic K-theory which was more accessible to calculations. A number of results have appeared showing that the approximation is as good as could have been hoped for, and that TC is indeed more approachable than K-theory. For an overview with references, see Madsen's survey article [9].

However, the cyclotomic trace $K \to TC$ is not an equivalence, and it is of interest to gauge the structural differences between K-theory and TC. This is a particularly appealing question in the setup of [3] where both functors are functors of exact categories. What K-theory lacks in computational accessibility, it makes up for by means of a number of fundamental theorems. Apart from the resolution theorem and cofinality proven in [3], the most important properties of Quillen's K-theory is localization and Devissage. Localization in this sense is false for THH for trivial reasons, but we will show that a version of the Devissage theorem still applies both to THH and TC. The picture is thus ($*$ means somewhat weakened):

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K-theory theorem	*TC*	*THH*
Cofinality	Yes	Yes
Resolution	Yes*	Yes*
Localization	No	No
Devissage	Yes*	Yes*
Respects finite products	Yes	Yes

Cofinality, resolution and finite products are treated in [3] for THH. The first and last via special homotopies (see [3, 1.5]), and so the proofs extend to TC. In the first section we show how resolution extends to TC. On the list thus remains the negative localization and the affirmative Devissage.

Theorem 1 (Devissage). *Let $\mathfrak{D}$ be a small abelian category, and let $\mathfrak{C}$ be a full subcategory closed under taking subobjects, quotient objects and finite products. Assume there is an endofunctor $F:\mathfrak{D}\to\mathfrak{D}$ respecting finite products, and natural transformations $\eta:F\to 1$ consisting of monomorphisms with quotients in $ob\mathfrak{C}$. Furthermore, assume that for every object $d\in ob\mathfrak{D}$ there is a number N such that F applied N times on d is the zero object, then*

$$TC(S\mathfrak{C};p)\simeq TC(S\mathfrak{D};p)$$

and likewise for THH.

This should be compared with the usual Devissage theorem in [10, Theorem 4]. Our version is somewhat limited as it applies only to filtrations coming from successive applications of the functor F, but this is good enough for our applications, namely to torsion modules of Dedekind domains, and in the proof of

Theorem 2 (Vanishing of nil-terms in TC). *Let $\mathfrak{C}\subseteq\mathfrak{D}$ be a resolving subcategory of an abelian category (see Definition 1.4). Assume that for any $d\in ob\mathfrak{D}$ there exist an exact sequence*

$$0\to c_n\to c_{n-1}\to\cdots\to c_1\to c_0\to d\to 0$$

in $\mathfrak{D}$ with all the $c_i\in ob\mathfrak{C}$. Then

$$TC(S(Nil(\mathfrak{C}));p)\simeq TC(S(Nil(\mathfrak{D}));p)\simeq TC(S\mathfrak{D};p)\simeq TC(S\mathfrak{C};p)$$

and likewise for THH.

In particular, this shows that the “nil-terms” vanish for TC and THH of a regular and coherent ring. Regular means that finitely presented modules have finite projective dimension, and coherent means that the finitely presented modules form an abelian category. (Another use of the term “regular” is that all finitely generated modules should have finite projective dimension, and this implies regular and coherent in our sense, but not conversely.)

The Localization theorem does not seem to have any good analog for topological cyclic homology. Even for rings of characteristic p the localization fiber sequence of K-theory does not remain a fiber sequence if we apply $TC(-;p)$ to it. However, we still have a splitting of TC of the projective line over a ring A, into two copies of $TC(A;p)$ as in [10].

Most of the results in this paper holds equally well for THH and TC, though we state them only for TC.

It is a pleasure to acknowledge the many helpful conversations with Marcel Bökstedt. This paper builds on [3], and many of my views on these matters were formed during the enjoyable cooperation with Randy McCarthy.

1. Preliminaries

For any linear category $\mathfrak{C}$, recall the definition of $THH(\mathfrak{C})$ [3, 1.3 and 1.2.3]. The topological Hochschild homology carries a cyclic structure and for any prime p we define the topological cyclic homology

$$TC(\mathfrak{C};p) = \underset{\overleftarrow{\mathscr{F}(p)}}{\operatorname{holim}} \, |THH(\mathfrak{C})|^{C_{p^n}},$$

where $\mathscr{F}(p)$ is a category whose objects are the powers of p, and with two commuting morphisms $R=\phi, F=i : p^{n+1} \to p^n$ (see e.g. [3, 1.5.3]).

For any exact category $\mathfrak{C}$, let $S\mathfrak{C}$ be Waldhausen's construction, [11], yielding a simplicial exact category whose homotopy groups are $K_{i-1}(\mathfrak{C})$, $i \geq 1$. Recall that for a split exact category $\mathfrak{C}$ we have equivalences

$$\Omega TC(S\mathfrak{C};p) \simeq TC(\mathfrak{C};p)$$

[3, Proposition 2.1.3], and that for a unital ring A we have equivalences

$$TC(\mathscr{P}_A;p) \simeq TC(A;p)$$

[3, Proposition 2.1.5].

An important feature of the model where we have incorporated the S construction is that it satisfies additivity: Let $d_i : S_2\mathfrak{C} \to S_1\mathfrak{C} = \mathfrak{C}$ for $i = 0,1,2$ be the face maps in the S construction (d_i sends $c_2 \rightarrowtail c_1 \twoheadrightarrow c_0$ to c_i), then $d_0 \times d_2 : S_2\mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ induces an equivalence

$$TC(SS_2\mathfrak{C};p) \xrightarrow[\sim]{d_0 \times d_2} TC(S\mathfrak{C};p) \times TC(S\mathfrak{C};p).$$

We give an easy example showing how this may be useful.

Flasque rings. A flasque additive category [8], is an additive category $\mathfrak{C}$ on which there exists an endofunctor $\tau:\mathfrak{C}\to\mathfrak{C}$ such that there is a natural isomorphism $\tau\cong\tau\oplus 1$.

Proposition 1.1. *If $\mathfrak{C}$ is a flasque additive category, then $TC(S\mathfrak{C};p)\simeq THH(S\mathfrak{C})\simeq *$.*

Proof. Let $\theta:\mathfrak{C}\to S_2\mathfrak{C}$ be the functor sending $c\in ob\mathfrak{C}$ to $c\to c\oplus\tau(c)\to\tau(c)$ (no twisting), where τ is an endofunctor making $\mathfrak{C}$ flasque. Consider the two composites

$$\begin{tikzcd} THH(S\mathfrak{C}) \arrow[r, "\theta"] & THH(SS_2\mathfrak{C}) \arrow[r, equal] \arrow[d, "d_0{,}d_2"'] & THH(SS_2\mathfrak{C}) \arrow[r, "d_1"] & THH(S\mathfrak{C}) \\ & THH(S\mathfrak{C})\times THH(S\mathfrak{C}), \arrow[ur, "\phi"'] & & \end{tikzcd}$$

where ϕ is the loop sum of the two maps induced by the two functors $\mathfrak{C}\to S_2\mathfrak{C}$ sending an object c to $c=c\to 0$ and $0\to c=c$. The upper composite is induced by $\tau\oplus 1\cong\tau$, whereas the lower is the sum of the maps induced by the identity and τ. The middle triangle commutes up to homotopy by additivity, and so we get that the identity on $THH(S\mathfrak{C})$ is homotopic to the constant map. Likewise for TC. $\square$

A ring A is said to be flasque if there is a nontrivial finitely generated projective module m such that $m\oplus A\cong A$. This is equivalent to $\mathscr{P}_A$ being flasque in the above sense [8].

Corollary 1.2. *If A is a flasque ring, then $TC(A;p)\simeq THH(A)\simeq *$.*

In particular, if A is any ring, the cone CA is flasque. The same is true if we choose an infinite-dimensional vector space V, and consider the ring $End(V)$.

The equivalence criterion. Let X be a finite pointed set. For any $c\in ob\mathfrak{C}$, we let $c\otimes\tilde{\mathbf{Z}}[X]$ simply mean the sum $\bigoplus_{x\in X-*}(c)$ of c with itself over all elements X different from the basepoint. Let $\mathscr{E}_X(\mathfrak{C})$ be the exact category with objects (c,v) with $c\in ob\mathfrak{C}$ and $v:c\to c\otimes\tilde{\mathbf{Z}}[X]$ a map in $\mathfrak{C}$ (or what amounts to the same thing: an endomorphism $v_x\in End(c)$ for each $x\in X$ different from the basepoint) (see [3, 2.3]). A morphism from (c,v) to (d,w) in $\mathscr{E}_X(\mathfrak{C})$ is an $f:c\to d\in\mathfrak{C}$ such that $(f\otimes id)v=wf$ (or equivalently $fv_x=w_xf$ for all $x\neq *$).

Then we have:

Lemma 1.3 (Equivalence criterion for TC). *Let $F:\mathfrak{C}\to\mathfrak{D}$ be an exact functor inducing a weak equivalence $S\mathscr{E}_X(\mathfrak{C})\to S\mathscr{E}_X(\mathfrak{D})$ for any finite pointed set X. Then it induces a weak equivalence*

$$TC(S\mathfrak{C};p)\to TC(S\mathfrak{D};p).$$

Proof. By the equivalence criterion for THH [3, 2.3.2] the conditions ensure that f induces a weak equivalence $THH(S\mathfrak{C})\to THH(S\mathfrak{D})$. As $TC(-;p)$ is a homotopy limit over the fixed point sets $|THH(-)|^{C_{p^n}}$, the lemma will follow if we know that f induces an equivalence $|THH(S\mathfrak{C})|^{C_{p^n}} \to |THH(S\mathfrak{D})|^{C_{p^n}}$ for each n. Recall that there is a natural equivalence between $\lim_{k\to\infty} \Omega^k |THH(\mathfrak{C}, S^k)|_{hC_{p^n}}$ and the fiber of the restriction map $|THH(\mathfrak{C})|^{C_{p^n}} \xrightarrow{R} |THH(\mathfrak{C})|^{C_{p^{n-1}}}$ (see [7, 2; 4, Lemma 11.2] and its proof, which applies equally well in our situation). Hence, the lemma follows by induction from the map of fiber sequences

$$\begin{tikzcd}
\lim_{k\to\infty} \Omega^k |THH(S\mathfrak{C}; S^k)|_{hC_{p^n}} \arrow[r] \arrow[d, "\simeq"'] & {|THH(S\mathfrak{C})|^{C_{p^n}}} \arrow[r, "R"] \arrow[d] & {|THH(S\mathfrak{C})|^{C_{p^{n-1}}}} \arrow[d] \\
\lim_{k\to\infty} \Omega^k |THH(S\mathfrak{D}; S^k)|_{hC_{p^n}} \arrow[r] & {|THH(S\mathfrak{D})|^{C_{p^n}}} \arrow[r, "R"] & {|THH(S\mathfrak{D})|^{C_{p^{n-1}}}}
\end{tikzcd}$$

and the fact that homotopy orbits preserve equivalences. □

The resolution theorem. The resolution theorem of [3] is a little too weak for our applications, and we now show how it can be extended.

Definition 1.4. We say that a full exact subcategory $\mathfrak{C} \subseteq \mathfrak{D}$ is resolving if

(1) it is closed under extensions,

(2) any map $c \twoheadrightarrow c'' \in \mathfrak{C}$ which is an admissible epimorphism in $\mathfrak{D}$ is an admissible epimorphism in $\mathfrak{C}$,

(3) given any object $d \in ob\mathfrak{D}$ and finite tuple of endomorphisms

$$\{t_i \in End_{\mathfrak{D}}(d), 0 < i < k\},$$

there exist commuting diagrams

$$\begin{tikzcd}
c \arrow[r, "s_i"] \arrow[d, "f"'] & c \arrow[d, "f"] \\
d \arrow[r, "t_i"] & d
\end{tikzcd} \qquad 0 < i < k,$$

with f an admissible epimorphism and $c \in ob\mathfrak{C}$.

Note that the condition (3) is automatically satisfied if for every $d \in ob\mathfrak{D}$ there exists an admissible epimorphism $f: c \to d$ with $c \in ob\mathfrak{C}$ projective in $\mathfrak{D}$.

Resolution Theorem 1.5. *Let $\mathfrak{C} \subseteq \mathfrak{D}$ be a resolving subcategory. Let $\mathfrak{C}_n$ be the full subcategory of $\mathfrak{D}$ consisting of objects d for which there exist an exact sequence*

$$0 \to c_n \to c_{n-1} \to \cdots \to c_1 \to c_0 \to d \to 0$$

in $\mathfrak{D}$ with all the $c_i \in ob\mathfrak{C}$. Set $\mathfrak{C}_\infty = \lim_{\to} \mathfrak{C}_n$. Then

$$TC(S\mathfrak{C};p) \simeq TC(S\mathfrak{C}_1;p) \simeq \cdots \simeq TC(S\mathfrak{C}_\infty;p)$$

and

$$THH(S\mathfrak{C}) \simeq THH(S\mathfrak{C}_1) \simeq \cdots \simeq THH(S\mathfrak{C}_\infty).$$

Proof. Consider the pair $\mathfrak{C}_n \subseteq \mathfrak{C}_{n+1}$. The "standard facts" [10, pp. 102–110] about this pair give that it is an inclusion of a resolving subcategory with the additional property that if $d \to c$ is an admissible monomorphism in $\mathfrak{C}_{n+1}$ with $c \in ob\mathfrak{C}_n$, then $d \in ob\mathfrak{C}_n$. Recall the proof of the resolution theorem in [3, 2.3.3]. The projectivity assumption listed in the statement is used rather weakly, and the given properties of $\mathfrak{C}_n \subseteq \mathfrak{C}_{n+1}$ are custom made to let the proof of [3, 2.3.3] work to prove that $S\mathscr{E}_X(\mathfrak{C}_n) \xrightarrow{\simeq} S\mathscr{E}_X(\mathfrak{C}_{n+1})$ for an arbitrary finite pointed set X. Using that K-theory commutes with filtered limits we get

$$S\mathscr{E}_X(\mathfrak{C}) \xrightarrow{\simeq} S\mathscr{E}_X(\mathfrak{C}_n) \xrightarrow{\simeq} S\mathscr{E}_X(\mathfrak{C}_{n+1}) \xrightarrow{\simeq} S\mathscr{E}_X(\mathfrak{C}_\infty),$$

which by the equivalence criterion gives the theorem. □

2. Devissage

In this section we prove the Devissage theorem, and list some immediate corollaries.

2.1. Proof of the Devissage Theorem 1. For every finite pointed set X, we must show that $\mathscr{E}_X(\mathfrak{C}) \subseteq \mathscr{E}_X(\mathfrak{D})$ fulfill the requirements in Quillen's Devissage theorem.

First, $\mathscr{E}_X(\mathfrak{D})$ is abelian. This is true as it obviously is an additive category and inherits kernels and cokernels from $\mathfrak{D}$. Secondly, the inclusion $\mathscr{E}_X\mathfrak{C} \to \mathscr{E}_X\mathfrak{D}$ is full and closed under formation of subobjects, quotient objects and finite products.

Thirdly, given an object $d \xrightarrow{f} d_X$ in $\mathfrak{D}$ we have a diagram

$$\begin{tikzcd}
 & F(d) \arrow[r, "\eta_k(d)"] \arrow[d, "F(f)"'] & d \arrow[d, "f"] \\
F(d)_X \arrow[r, "\cong"] & F(d_X) \arrow[r, "\eta_k(d_X)"] & d_X
\end{tikzcd}$$

giving a finite filtration

$$0 = F(F(\ldots F(f)\ldots)) \subset \cdots \subset F(F(f)) \subset F(f) \subset f,$$

which is a filtration with cokernels in $\mathscr{E}_X\mathfrak{C}$. □

Corollary 2.2. *Given any Dedekind domain A, let $\mathfrak{T}$ be the category of finitely generated torsion modules and $\mathfrak{S} \subseteq \mathfrak{T}$ the full subcategory of semi-simple modules, then*

$$\Omega TC(S\mathfrak{T}; p) \simeq \Omega TC(S\mathfrak{S}; p) \simeq TC(\mathfrak{S}; p),$$

and likewise for THH.

Proof. The last equivalence follows from the fact that $\mathfrak{S}$ is split exact. For the first equivalence we are in the situation where we can apply the Devissage theorem. For any object in $t \in \mathfrak{T}$ and maximal ideal m consider the map $t \to t/m$. Only finitely many t/m are nonzero, so $t \to \prod_{m \in \mathrm{Max}(A)} t/m$ is a well-defined morphism in $\mathfrak{T}$ with target in $\mathfrak{S}$. This is functorial, and we let the functor F in the Devissage theorem simply be the kernel. □

Corollary 2.3. $\Omega THH(S\mathfrak{T}) \simeq \prod'_{m \in \mathrm{Max}(A)} THH(A/m)$ *(the weak product).*

Proof. This follows from the corollary above as THH respects finite products and colimits. □

Corollary 2.4. *Let A be a Dedekind domain, and assume either that $\mathrm{char}\, A \neq p$ or that A has only finitely many maximal ideals. Then*

$$\Omega TC(S\mathfrak{T}; p)^{\widehat{}}_p \simeq \prod_{m \in \mathrm{Max}(A)} TC(A/m; p)^{\widehat{}}_p$$

(the product actually only has finitely many noncontractible factors).

Proof. The case where A has only finitely many maximal ideals is clear as TC respects finite products.

So, assume $\mathrm{char}\, A \neq p$, and let $\mathfrak{S}_p$ and $\mathfrak{S}_p^{\perp}$ be the full subcategories of $\mathfrak{S}$ of p-torsion and uniquely p-divisible modules. As any module in $\mathfrak{S}$ is of the form $\prod_{i=0}^{N} (A/m_i)^{n_i}$, where m_i are distinct maximal ideals, and so each factor is either p-torsion or uniquely p-divisible, we get that $\mathfrak{S} = \mathfrak{S}_p \times \mathfrak{S}_p^{\perp}$. Thus, $TC(S\mathfrak{T}; p) \simeq TC(S\mathfrak{S}_p; p) \times TC(S\mathfrak{S}_p^{\perp}; p)$. But the fact that each *Hom* set in $S\mathfrak{S}_p^{\perp}$ is uniquely p-divisible implies that the homotopy groups of $THH(S\mathfrak{S}_p^{\perp})$ are uniquely p-divisible, and by the argument of Goodwillie in [4, 13] we get that $\operatorname{holim}_{\overleftarrow{\mathscr{F}(p)}} |THH(S\mathfrak{S}_p^{\perp})^{C_{p^n}}| \to |THH(S\mathfrak{S}_p^{\perp})|$ is an equivalence, and so after p-completion both spaces vanish. Thus, $TC(S\mathfrak{S}_p^{\perp}; p)^{\widehat{}}_p \simeq *$.

As A is a Dedekind domain, there are only finitely many maximal ideals that contain pA, and so any element in $\mathfrak{S}_p$ can be uniquely written as a product $\prod_{m \in M_p} (A/m)^{n_m}$ where M_p is the set of maximal ideals containing pA. Hence

$$\Omega TC(S\mathfrak{S}_p; p)^{\widehat{}}_p \simeq \prod_{m \in M_p} TC(A/m; p)^{\widehat{}}_p.$$

As to the missing factors, we note that they all are contractible as they are TC of fields of characteristic different from p. $\square$

Corollary 2.2 is due to Marcel Bökstedt (my original proof was only for the case with countably many maximal ideals). He also pointed out to me that the product in Corollary 2.4 might be finite.

3. On the fundamental theorem

A version of the fundamental theorem of K-theory states that $\Omega K(A[t]) \times K(A) \simeq \Omega K(A) \times K(Nil\,\mathscr{P}_A)$. It is interesting to investigate to what extent this is true for TC and THH. If A is regular and coherent then $K(Nil\,\mathscr{P}_A) \simeq K(A)$, and so the statement reads $K(A[t]) \simeq K(A)$. It was noted in [6] that this formula is false for TC (it is trivially false for THH). One could conceive that this failure was due to some nonvanishing $TC(Nil\,\mathscr{P}_A)$, but Theorem 2 rules that out, and TC really is very different from K-theory in this respect.

The vanishing of the Nil terms in K-theory relies on two facts: (1) By Devissage $K(Nil(\mathfrak{C})) \simeq K(\mathfrak{C})$ if $\mathfrak{C}$ is abelian. (2) Resolutions in $\mathfrak{C}$ give rise to resolutions in $Nil(\mathfrak{C})$, and so the resolution theorem applies to prove that in case A is regular and coherent $K(Nil\,(\mathscr{P}_A)) \simeq K(Nil\,(\mathscr{M}_A)) \simeq K(\mathscr{M}_A) \simeq K(A)$, where $\mathscr{M}_A$ is the category of finitely presented A-modules.

The problem with this approach, is that whereas $\mathscr{M}_A$ have finite projective resolutions, the resolutions derived from this in $Nil\,(\mathscr{M}_A)$ are not projective. K-theory cares nothing about this, but the resolution theorem for TC and THH is slightly weaker so care is needed. In the case where A is a field $\mathscr{M}_A \cong \mathscr{P}_A$ and so the Nil-terms must always vanish, and this is in accordance with the statements for $TC(\mathbf{F}_p; p)$ given in [6].

We will now show Theorem 2 by handicrafting resolutions in $\mathscr{E}_X(Nil\,\mathfrak{C})$ (following Bass [1]) that meet the sharpened version of the resolution theorem.

Proof of Theorem 2. Consider

$$\begin{tikzcd} d \arrow[r, "t"] \arrow[d, "\beta"'] & d_X \arrow[d, "\beta_X"] \\ d \arrow[r, "t"] & d_X \end{tikzcd} \quad \in ob\mathscr{E}_X(Nil\,\mathfrak{D}),$$

where $\beta^n = 0$, and let $f : c \twoheadrightarrow d$ be an epimorphism with $c \in ob\mathfrak{C}$ equipped with a map $s : c \to c_X$ such that $f_X s = tf$. Let $F : c^n \twoheadrightarrow d$ be given by $F \circ in_j = \beta^{j-1} f$ for $1 \leq j \leq n$ ($\beta^0 = id$). Let $\alpha : c^n \to c^n$ be given by $\alpha_{i,j} = \delta_{i-1,j}$ for $1 \leq i,j \leq n$. Note that $\alpha^n = 0$.

Finally, let $\sigma : c^n \to (c^n)_X = (c_X)^n$ be $s : c \to c_X$ on each coordinate. Then

$$\begin{tikzcd}
& c^n \arrow[rr, "\sigma"] \arrow[dl, "\alpha"'] \arrow[dd, "F" near start] & & c^n_X \arrow[dl, "\alpha_X"] \arrow[dd, "F_X"] \\
c^n \arrow[rr, "\sigma" near start] \arrow[dd, "F"'] & & c^n_X \arrow[dd, "F_X" near start] & \\
& d \arrow[rr, "t" near start] \arrow[dl, "\beta"] & & d_X \arrow[dl, "\beta_X"] \\
d \arrow[rr, "t"] & & d_X &
\end{tikzcd}$$

commutes. We check the front square

$$(tF) \circ in_j = t\beta^{j-1} f = \beta_X^{j-1} t f = \beta_X^{j-1} f_X s = (F_X \sigma) \circ in_j.$$

The other squares are no more difficult. This shows that $Nil(\mathfrak{C}) \subseteq Nil(\mathfrak{D})$ satisfy the requirements in the resolution theorem, and so $TC(S(Nil(\mathfrak{C})); p) \simeq TC(S(Nil(\mathfrak{D})); p)$ and likewise for THH.

So it just remains to prove that the Nil terms vanish for any abelian category. But this follows from the Devissage theorem, for let $\mathfrak{D} \subseteq Nil(\mathfrak{D})$ be considered as a subcategory under the inclusion $d \mapsto (d, 0)$. Let $F : Nil(\mathfrak{D}) \to Nil(\mathfrak{D})$ be the endofunctor sending an object (d, β) to $(im\beta, \beta)$ and η just the inclusion $F \subseteq 1$. All the quotients are in $\mathfrak{D}$, and the filtration is finite, and so $TC(S(Nil(\mathfrak{D})); p) \simeq TC(S\mathfrak{D}; p)$ and $THH(S(Nil(\mathfrak{D}))) \simeq THH(S\mathfrak{D})$. □

Corollary 3.1. *Let A be a regular and coherent ring. Then $\Omega TC(S(Nil(\mathscr{P}_A)); p) \simeq TC(A; p)$.*

It is worthwhile to note that the resolutions in $Nil(\mathfrak{D})$ are not projective. For let $c \xrightarrow{0} c \in Nil(\mathfrak{C})$ where c is projective in $\mathfrak{D}$. This is not projective in $Nil(\mathfrak{D})$ for there is no map $c \to c^2$ splitting the square

$$\begin{tikzcd}
c^2 \arrow[r, "\binom{00}{10}"] \arrow[d, "pr_1"'] & c^2 \arrow[d, "pr_1"] \\
c \arrow[r, "0"] & c.
\end{tikzcd}$$

Localization. Corollary 3.1 shows that we cannot have a "localization sequence" for $TC(-;p)$, even if the relevant prime stays noninvertible. More precisely, let A be a ring and let $S \subset A$ be some multiplicatively closed subset of central nonzero divisors. Let $\mathscr{H}_A^S$ be the category of A modules P of projective dimension ≤ 1 and such that $S^{-1}P = 0$. Then there is a fibration sequence

$$S\mathscr{H}_A^S \to S\mathscr{P}_A \to S\mathscr{P}_{S^{-1}A}. \tag{$*$}$$

Example 3.2. Let $A = \hat{\mathbf{Z}}_p[t]$ and $S^{-1}A = \hat{\mathbf{Z}}_p[t, t^{-1}]$. The localization sequence $(*)$ is not a fiber sequence after applying TC to it.

To see this, note that Hesselholt [6, p. 10ff.] shows that there is a space, say $V(A)$, such that the fiber of $TC(A[t]; p)\hat{}_p \to TC(A[t,t^{-1}]; p)\hat{}_p$ is equivalent to $TC(A;p)\hat{}_p \times \Omega V(A)$. Furthermore, he shows that the homotopy groups $\pi_i(V(\hat{\mathbf{Z}}_p); \mathbf{F}_p)$ are highly nontrivial for $2 \leq i \leq 2p-3$. But $Nil(\mathscr{P}_A) \to \mathscr{H}_{A[t]}^{(t)}$ sending (P, f) to coker $\{P[t] \xrightarrow{t-f} P[t]\}$ is a well-defined equivalence of categories [5, p. 263], so $TC(S\,Nil(\mathscr{P}_A); p) \simeq TC(S\mathscr{H}_{A[t]}^{(t)}; p)$. If A is regular and coherent this shows that $\Omega TC(S\mathscr{H}_{A[t]}^{(t)}; p) \simeq \Omega TC(S\mathscr{P}_A;\ p) \simeq TC(A;p)$, and, in particular, that $\Omega TC(S\mathscr{H}_{\hat{\mathbf{Z}}_p[t]}^{(t)}; p)\hat{}_p$ is not equivalent to the fiber of $TC(\hat{\mathbf{Z}}_p[t]; p)\hat{}_p \to TC(\hat{\mathbf{Z}}_p[t,t^{-1}]; p)\hat{}_p$. Hesselholt has informed me that similar breakdown occurs in characteristic p, e.g. for polynomial rings with many variables.

TC of the projective line. When Grayson proves the fundamental theorem of K-theory in [5] he uses localization twice, plus a computation by Quillen of the K-theory of the projective line. That is, he considers the map of (localization) fibration sequences

$$\begin{tikzcd}
S\,Nil(\mathscr{P}_A) \arrow[r] \arrow[d, equal] & S\mathscr{P}_X \arrow[r] \arrow[d] & S\mathscr{P}_{A[t^{-1}]} \arrow[d] \\
S\,Nil(\mathscr{P}_A) \arrow[r] & S\mathscr{P}_{A[t]} \arrow[r] & S\mathscr{P}_{A[t,t^{-1}]}
\end{tikzcd},$$

where $\mathscr{P}_X \cong \mathscr{P}_{A[t]} \prod_{\mathscr{P}_{A[t,t^{-1}]}} \mathscr{P}_{A[t^{-1}]}$ serves as the category of vector bundles on the projective line, whose K-theory Quillen proves is twice $K(A)$.

It is interesting that the part pertaining to the projective line carries through in our setup.

Proposition 3.3. *With the notation above,*

$$\Omega TC(S\mathscr{P}_X; p) \simeq TC(A;p) \times TC(A;p).$$

Proof. Analyzing the proof of Quillen in [10, pp. 138–143], we see that the only facts about K-theory he uses are additivity for characteristic filtration, and that it commutes with filtered colimits. This is available for THH, yielding equivalences on THH between the relevant categories, and by the proof of Lemma 1.3 (no colimits being "left outside") also on TC. □

References

[1] H. Bass, Algebraic K-Theory, Benjamin, New York, 1968, pp. 268–269, 448–451, 522–529

[2] B.I. Dundas, R. McCarthy, Stable K-theory and topological Hochschild homology, Ann. of Math. 140 (1994), 685–701.

[3] B.I. Dundas, R. McCarthy, Topological Hochschild homology of ring functors and exact categories, J. Pure Appl. Algebra 109 (1996) 231–294.

[4] T.G. Goodwillie, Notes on the cyclotomic trace, 23 March 1990, Lecture notes from a lecture series given at MSRI, Spring 1990.

[5] D. Grayson, (after Quillen), Higher algebraic K-theory II, Lecture Notes in Mathematics, vol. 551, Springer, Berlin, 1976, pp. 217–240.

[6] L. Hesselholt, Topological cyclic homology and local function fields, Aarhus Universitet, Preprint Series, December 1993, no. 31.

[7] L. Hesselholt, I. Madsen, On the K-theory of finite algebras over Witt vectors of perfect fields, Topology 36 (1997) 29–101.

[8] M. Karoubi, O. Villamajor, K-théorie algébrique et K-théorie topologique, Math. Scan. 28 (1971) 265–307.

[9] I. Madsen, Algebraic K-theory and traces, in: Current Developments in Mathematics, International Press, 1995, pp. 191–323.

[10] D. Quillen, Higher Algebraic K-Theory: I, Lecture Notes in Mathematics, vol. 341, Springer, Berlin, 1973, pp 77–139.

[11] F. Waldhausen, Algebraic K-theory of spaces, in: Algebraic and Geometric Topology, Proc. Rutgers 1983, Lecture Notes in Mathematics, vol. 1126, Springer, Berlin, 1985, pp. 318–419.